INVERSE DESCENT STATISTIC FOR ANDRÉ AND SIMSUN PERMUTATIONS

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ABSTRACT. Simsun permutations, André I permutations and André II permutations are three combinatorial models for Euler numbers. It's known that the descent statistic is equidistributed over the set of André I permutations and the set of simsun permutations. In this paper, we prove that the trivariate statistic (ides, des, maj), comprising the inverse descent, descent, and major index, are equidistributed over these three sets. This result is equivalent to showing that the inverse descent is equidistributed over these three sets that share the same tree shape. The proof of the equidistribution of the inverse descent over the set of André I permutations and the set of André II permutations with the same tree shape reduces to establishing new refinements of Stanley's shuffle theorem.

1. INTRODUCTION

In the field of enumerative combinatorics, several kinds of permutations are counted by *Euler numbers*, such as *alternating permutations*, *André I and II permutations*, and *simsun permutations*. *Euler numbers*, denoted by E_n, are a sequence of integers that arise in the Taylor series expansions of $\sec(x) + \tan(x)$. Their combinatorial significance was cemented by the work of André in the late 19th century [2]. André proved that E_n counts the number of *alternating permutations* of length n (see [28]), which are permutations $\sigma = \sigma_1\sigma_2 \dots \sigma_n$ satisfying $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$.

André permutations were first introduced by Foata and Schützenberger and further studied by Strehl [30] and Foata and Strehl [12, 13]. For clarity, we will work with permutations of length n for which each permutation is a sequence of n distinct integers not necessarily from 1 to n. The empty word e and any single-letter word are defined as both *André I permutations* and *André II permutations*. For a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ $(n \geq 2)$ of length n, we decompose it as $\sigma = \tau \min(\sigma) \tau'$. Here σ is the concatenation of a left factor τ, followed by the minimum letter $\min(\sigma)$, and a right factor τ'. Then, σ is called an *André I permutation* (resp. *André II permutation*) if both τ and τ' are André I permutations (resp. André II permutations),

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and the maximum letter of the subword $\tau\tau'$ lies in τ' (resp. the minimum letter of $\tau\tau'$ lies in τ').

The set of all André I permutations on the set $[n] := \{1, 2, \ldots, n\}$ is denoted by And_n^{I} and the set of André II permutations on the set $[n]$ is denoted by And_n^{II}. This inductive definition immediately reveals a connection to the Euler numbers, as it can be shown that the number of André I permutations and André II permutations on the set $[n]$ are equal, i.e., $E_n = |\mathrm{And}_n^{I}| = |\mathrm{And}_n^{II}|$.

André I permutations for $n \le 5$ are listed below:

$n = 1$: 1; $n = 2$: 12; $n = 3$: 123, 213;
$n = 4$: 1234, 1324, 2314, 2134, 3124;
$n = 5$: 12345, 12435, 13425, 23415, 13245, 14235, 34125, 24135, 23145, 21345, 41235, 31245, 21435, 32415, 41325, 31425.

André II permutations for $n \le 5$ are listed below:

$n = 1$: 1; $n = 2$: 12; $n = 3$: 123, 312;
$n = 4$: 1234, 1423, 3412, 4123, 3124;
$n = 5$: 12345, 12534, 14523, 34512, 15234, 14235, 34125, 45123, 35124, 51234, 41235, 31245, 51423, 53412, 41523, 31524.

Simsun permutations were introduced by Rodica Simion and Sheila Sundaram in a series of studies of homology representations of the symmetric group [31, 32]. To better elaborate on our results, we adopt the following definition of simsun permutations. A permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ on the set $[n]$ is called a simsun permutation if $\sigma_n = n$ and it contains no double descents, and this property is preserved after removing the elements $n, n-1, n-2, \ldots, 1$ in order. For example, it is easy to see that $\sigma = 21473658$ is a simsun permutation since 21473658, 2147365, 214365, 21435, 2143, 213, 21, 1 have no double descents. Recall that an index i (where $1 \le i < n$) is called a *descent* of a permutation $\sigma = \sigma_1 \ldots \sigma_n$ if $\sigma_i > \sigma_{i+1}$ and an index i (where $1 \le i \le n-2$) is called a *double descent* if $\sigma_i > \sigma_{i+1} > \sigma_{i+2}$.

Notably, if one removes the last element from a simsun permutation as defined here, the resulting permutation aligns with the original definition of simsun permutations due to Simion and Sundaram.

The set of all simsun permutations on the set $[n]$ is denoted by RS_n. A remarkable property of simsun permutations is that $|\mathrm{RS}_n| = E_n$. The notation RS_n was first adopted by Chow and Shiu [7].

Simsun permutations for $n \le 5$ are listed below:

$n = 1$: 1; $n = 2$: 12; $n = 3$: 123, 213;
$n = 4$: 1234, 1324, 2134, 2314, 3124;
$n = 5$: 12345, 12435, 13245, 13425, 14235, 21345, 21435, 23145, 23415, 24135, 31245, 31425, 34125, 41235, 41325, 42315.

André permutations and simsun permutations provide new combinatorial interpretations for the Euler numbers. They play an important role in the study of *cd*-indices of simplicial Eulerian posets. For results along this line, please see [5, 6, 17, 18, 22, 23, 27]. Other properties about André permutations and simsun permutations have been extensively studied by Barnabei et al. [4], Chow and Shiu [7], Deutsch-Elizalde [8], Disanto [9], Foata and the first author [11] and so on. In particular, by constructing a bijection between the set of André I permutations and the set of simsun permutations, Chow and Shiu [7] observed that the number of descents are equidistributed over André I permutations and simsun permutations. Specifically, let $\mathrm{des}(\sigma)$ denote the number of descents of σ, they showed that

$$\sum_{\sigma\in\mathrm{And}_n^I} t^{\mathrm{des}(\sigma)} = \sum_{\sigma\in\mathrm{RS}_n} t^{\mathrm{des}(\sigma)}.$$

In this paper, we show that the number of inverse descents are also equidistributed over André permutations and simsun permutations. The number of inverse descents of a permutation σ is simply the number of descents of its inverse permutation σ^{-1}, namely, $\mathrm{ides}(\sigma)=\mathrm{des}(\sigma^{-1})$. In fact, we show that the trivariate statistic (ides, des, maj) are equidistributed over André permutations and simsun permutations, where *the major index* $\mathrm{maj}(\sigma)$ of σ is defined to be the sum of its descents of σ. For brevity, we adopt the notation n-André permutations for André permutations on $[n]$ and n-simsun permutations for simsun permutations on $[n]$.

Our main result is as follows.

Theorem 1. *The trivariate statistic* (ides, des, maj) *are equidistributed over the set of n-André I permutations, n-André II permutations and n-simsun permutations, i.e.,*

$$\begin{aligned}\sum_{\sigma\in\mathrm{And}_n^I} s^{\mathrm{ides}(\sigma)}t^{\mathrm{des}(\sigma)}q^{\mathrm{maj}(\sigma)} &= \sum_{\sigma\in\mathrm{And}_n^{II}} s^{\mathrm{ides}(\sigma)}t^{\mathrm{des}(\sigma)}q^{\mathrm{maj}(\sigma)}\\ &= \sum_{\sigma\in\mathrm{RS}_n} s^{\mathrm{ides}(\sigma)}t^{\mathrm{des}(\sigma)}q^{\mathrm{maj}(\sigma)}.\end{aligned}$$

To our knowledge, even the special case of the above result for the univariate statistic "ides" is new:

$$A_n(s) := \sum_{\sigma\in\mathrm{And}_n^I} s^{\mathrm{ides}(\sigma)} = \sum_{\sigma\in\mathrm{And}_n^{II}} s^{\mathrm{ides}(\sigma)} = \sum_{\sigma\in\mathrm{RS}_n} s^{\mathrm{ides}(\sigma)}.$$

We list the first values of the polynomials $A_n(s)$ below:

$$\begin{aligned}&A_1(s)=1,\quad A_2(s)=1,\quad A_3(s)=s+1,\quad A_4(s)=4s+1,\\ &A_5(s)=4s^2+11s+1,\quad A_6(s)=2s^3+32s^2+26s+1.\end{aligned}$$

The proof of Theorem 1 can be sketched as follows: By sending André permutations and simsun permutations to increasing binary trees and applying Proposition 5 in Section 2, the proof of Theorem 1 reduces to showing that the inverse descents (ides) is equidistributed over the n-André I permutations, the n-André II permutations and the n-simsun permutations that share the same tree shape (see Theorem 7 in Section 2). The proof of Theorem 7 is split into two parts: (a) proving the equidistribution of ides over n-André I permutations and n-André II permutations with the same tree shape (relation (a) in Theorem 7), and (b) proving the equidistribution of ides over the n-André II permutations and the simsun permutations with the same tree shape (relation (b) in Theorem 7). Specially, the proof of relation (a) relies on an investigation of the shuffle of permutations (see Section 3 for details), while the proof of relation (b) proceeds by constructing a bijection between the set of n-André II permutations and the set of the n-simsun permutations (see Section 4). It would be interesting to give a direct explicit bijective proof of relation (a).

In the proof of relation (a) in Theorem 7, the following general result plays an important role. Note that this result applies to ordinary permutations, not solely André permutations.

Let $\mathfrak{S}_n$ denote the set of permutations on the set $[n]$. Suppose that $\sigma \in \mathfrak{S}_j$ and $\tau \in \mathfrak{S}_k$. We define the following three sets of permutations:

$$\begin{aligned} \sigma \Diamond \tau &= \{\mu = \sigma' 1 \tau' \in \mathfrak{S}_{j+k+1} \mid \sigma' \sim \sigma,\ \tau' \sim \tau\}; \\ \sigma \vartriangle \tau &= \{\mu = \sigma' 1 \tau' \in \mathfrak{S}_{j+k+1} \mid \sigma' \sim \sigma,\ \tau' \sim \tau,\ j+k+1 \in \tau'\}; \\ \sigma \triangledown \tau &= \{\mu = \sigma' 1 \tau' \in \mathfrak{S}_{j+k+1} \mid \sigma' \sim \sigma,\ \tau' \sim \tau,\ 2 \in \tau'\}, \end{aligned}$$

where $\sigma' \sim \sigma$ means that reducing the letters of σ' to $\{1, 2, \ldots, j\}$ yields σ.

For example, if $\mu = \sigma' 1 \tau' = 692581473$, then $\sigma' = 69258$, which reduces to $\sigma = 35124$, $\tau' = 473$ which reduces to $\tau = 231$.

Theorem 2. *Let $\sigma \in \mathfrak{S}_j$ and $\tau \in \mathfrak{S}_k$ be two permutations with* $\mathrm{ides}(\sigma) = j'$ *and* $\mathrm{ides}(\tau) = k'$. *Then*

$$\sum_{\mu \in \sigma \Diamond \tau} t^{\mathrm{ides}(\mu)} = \sum_{i \geq 1} \binom{k - k' + j' + 1}{i + j' - k'} \binom{j - j' + k' - 1}{i - 1} t^{i + j'}, \tag{1}$$

$$\sum_{\mu \in \sigma \vartriangle \tau} t^{\mathrm{ides}(\mu)} = \sum_{i \geq 1} \binom{k - k' + j'}{i + j' - k'} \binom{j - j' + k' - 1}{i - 1} t^{i + j'}, \tag{2}$$

$$\sum_{\mu \in \sigma \triangledown \tau} t^{\mathrm{ides}(\mu)} = \sum_{i \geq 1} \binom{k - k' + j'}{i + j' - k'} \binom{j - j' + k' - 1}{i - 1} t^{i + j'}. \tag{3}$$

Remark. When $t = 1$, we derive the following two identities, which also follow from the Chu-Vandermonde identity [29, Example 1.1.17]:

$$\begin{aligned} \binom{j+k}{j} &= \sum_{i \geq 1} \binom{j - d - 1}{i - 1} \binom{k + d + 1}{k - i + 1}, \\ \binom{j + k - 1}{j} &= \sum_{i \geq 1} \binom{j - d - 1}{i - 1} \binom{k + d}{k - i}. \end{aligned}$$

By considering the inverses of permutations, Theorem 2 can be transformed to special cases of three refinements of Stanley's shuffle theorem (see Theorem 12). Stanley's shuffle theorem was first established by Stanley [26] in his study of P-partitions. As observed by Gessel and Zhuang [14], this theorem implies that the major index (maj) and descent number (des) are shuffle compatible, which has motivated several recent works, including those by Adin, Gessel, Reiner and Roichman [1], Baker-Jarvis and Sagan [3], Domagalski, Liang, Minnich and Sagan [10], Grinberg [16], the second author and Zhang [19] and Yang and Yan [33]. Bijective proofs of Stanley's Shuffle Theorem have been given by Goulden [15], the second author and Zhang [20] and Stadler [24]. In particular, the second author and Zhang [20] established several refinements of this theorem based on their bijections. The proof of Theorem 2 in this paper also relies on their bijection, see Section 4 for more details.

2. Increasing binary trees

In this section, we aim to demonstrate that the proof of Theorem 1 is equivalent to proving Theorem 7 with the aid of the description of André permutations and simsun permutations in terms of increasing binary trees.

A *binary tree* is a rooted tree in which every vertex has either (i) no children, (ii) a single left child, (iii) a single right child, or (iv) both a left child and a right child. Vertices without children are called *leaves*, while all others are *internal* vertices. An *increasing binary tree* on the set $[n]$ is a binary tree with n vertices labeled $1, 2, \ldots, n$ such that the labels along any path from the root are increasing.

It is well known that there exists a bijection Ψ between the set of permutations on $[n]$ and the set of increasing binary trees on $[n]$, see [29, Chapter 1]. More precisely,

Definition 3 (The map Ψ)**.** *Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a sequence of n distinct letters not necessarily from 1 to n. Define a binary tree T_π as follows. If $\pi = \emptyset$, then $T_\pi = \emptyset$. If $\pi \neq \emptyset$, then let i be the least letter of π. Thus π can be factored uniquely in the form $\pi = \sigma i \tau$. Now let i be the root of T_π, and let T_σ and T_τ be the left and right subtrees obtained by removing i (see Figure 1). This yields an inductive definition of T_π.*

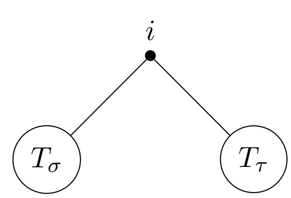

FIGURE 1. An inductive definition of T_π

As observed by Foata and the first author [11] and Chow and Shiu [7], when restricted to André permutations and simsun permutations, the bijection Ψ induces a bijection sending André permutations and simsun permutations to special cases of increasing binary trees, which we refer to as André trees and simsun trees, respectively.

Given an increasing binary tree T and a vertex s of T, let $T(s)$ denote the subtree of T with the root s, and let $T_l(s)$ and $T_r(s)$ denote the left and right subtrees rooted at s, respectively.

- An increasing binary tree T is said to be an *André I tree* if for any internal vertex s, the right subtree $T_r(s)$ contains the vertex of the maximum label in $T(s)$. By convention, the maxima of an empty subtree is defined as 0.
- An increasing binary tree T is said to be an *André II tree* if for any internal vertex s, the right subtree $T_r(s)$ contains the vertex with the minimum label in $T(s)$ excluding s itself. By convention, the minima of an empty subtree is defined as $+\infty$.
- An increasing binary tree on $[n]$ is called a *simsun tree* if n is its rightmost vertex, and when the vertices $n, n-1, n-2, \ldots, 1$ are removed in sequence, the resulting trees T' satisfy the following property: for any internal vertex s in a left subtree of T', if $T'_l(s)$ (the left subtree of T' with the root s) is non-empty, then $T'_r(s)$ (the right subtree of T' with the root s) is also non-empty.

Similarly, for brevity, we adopt the notation n-André trees for André trees on $[n]$ and n-simsun trees for simsun trees on $[n]$.

Proposition 4 ([7, 11])**.** *There exists a bijection Ψ between the set of n-André I permutations* (*resp. n-André II permutations, n-simsun permutations*) *and the set of n-André I trees* (*resp. n-André II trees, n-simsun trees*)*.*

Fig. 2 depicts a bijection between the set of 4-André I permutations and the set of 4-André I trees, while Fig. 3 shows a bijection between the set of 4-André II permutations and the set of 4-André II trees. Fig. 4 illustrates a bijection between the set of 4-simsun permutations and the set of 4-simsun trees.

The *shape* of a labeled tree T refers to its underlying unlabeled tree, denoted by shape(T). From the construction of the bijection Ψ, it is not difficult to see that the descent set of σ is determined by the shape of the tree corresponding to σ under the bijection Ψ. For a permutation $\sigma = \sigma_1 \cdots \sigma_n$, its descent set is defined as

$$\mathrm{Des}(\sigma) = \{1 \le i \le n-1 \colon \sigma_i > \sigma_{i+1}\}.$$

Then its descent number $\mathrm{des}(\sigma)$ is given by

$$\mathrm{des}(\sigma) := |\,\mathrm{Des}(\sigma)|.$$

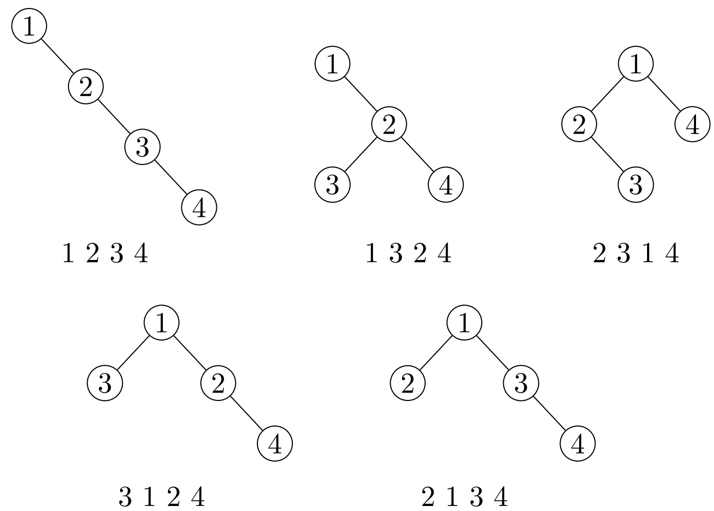

FIGURE 2. The bijection between 4-André I trees and 4-André I permutations

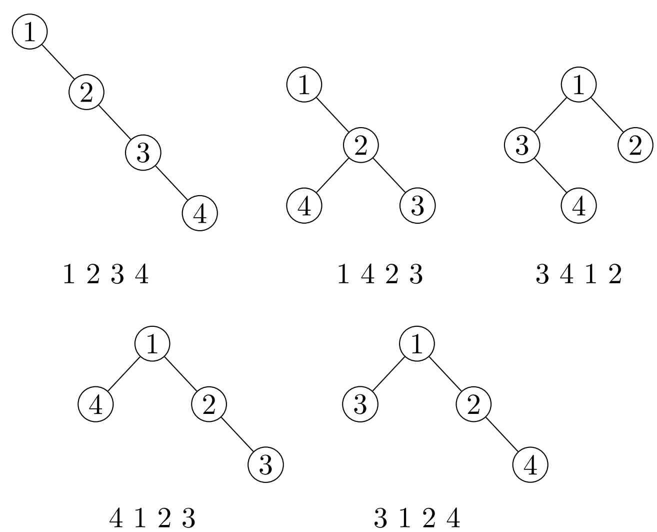

FIGURE 3. The bijection between 4-André II trees and 4-André II permutations

and its major index $\mathrm{maj}(\sigma)$ is given by

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i.$$

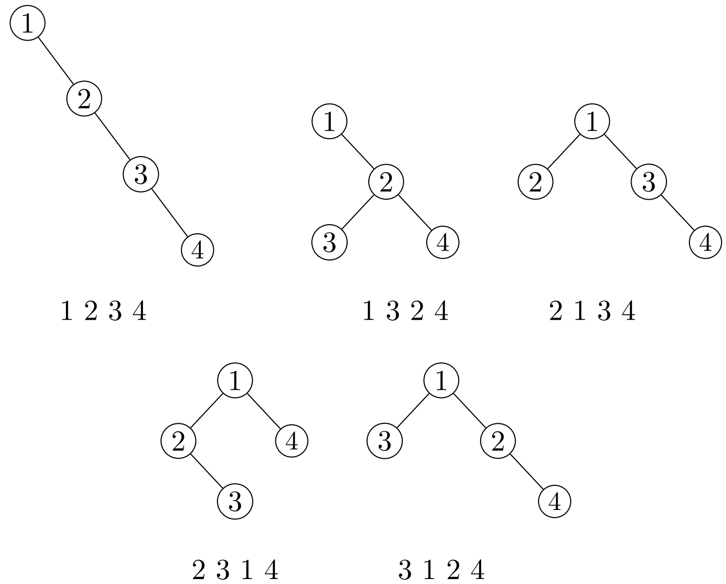

FIGURE 4. The bijection between 4-simsun trees and 4-simsun permutations

Proposition 5. *Let σ be a permutation, and let $T_\sigma = \Psi(\sigma)$ be the increasing binary tree corresponding to σ under the bijection Ψ. Then the descent set* $\mathrm{Des}(\sigma)$ *of σ is completely determined by the shape of T_σ.*

Note that the shape of T_σ in Proposition 5 is also referred to as the tree shape of the permutation σ. Let U_n^{RL} denote the set of the rooted unlabeled binary trees with n vertices in which no internal vertex has only a left child. It is not difficult to show that $|\,\mathrm{U}_n^{RL}\,| = M_n$, where M_n is the n-th Motzkin number defined by

$$M(x) = 1 + xM(x) + x^2M(x) = \sum_{n\geq 0} M_n x^n.$$

From the definitions of André trees and simsun trees, it is easy to verify that the tree shape of André permutations and simsun permutations belong to the set U_n^{RL}, that is,

Proposition 6. *Let σ be a permutation, and let $T_\sigma = \Psi(\sigma)$ be the increasing binary tree corresponding to σ under the bijection Ψ. Then*

$$\begin{aligned}\mathrm{U}_n^{RL} &= \{\mathrm{shape}(T_\sigma) \colon \sigma \in \mathrm{And}_n^{I}\}\\ &= \{\mathrm{shape}(T_\sigma) \colon \sigma \in \mathrm{And}_n^{II}\}\\ &= \{\mathrm{shape}(T_\sigma) \colon \sigma \in \mathrm{RS}_n\}.\end{aligned}$$

Given an unlabeled binary tree T in U_n^{RL}, let $\mathrm{And}^I(T)$ (resp. $\mathrm{And}^{II}(T)$, $\mathrm{RS}(T)$) denote the set of n-André I permutations (resp. n-André II permutations, n-simsun permutations) with tree shape T. Combining Proposition 5 and Proposition 6, we see that the proof of Theorem 1 is equivalent to establishing the following result:

Theorem 7. *For $n \geq 1$ and any unlabeled binary tree $T \in \mathrm{U}_n^{RL}$, we have*

$$\sum_{\sigma \in \mathrm{And}^I(T)} s^{\mathrm{ides}(\sigma)} \overset{(a)}{=} \sum_{\sigma \in \mathrm{And}^{II}(T)} s^{\mathrm{ides}(\sigma)} \overset{(b)}{=} \sum_{\sigma \in \mathrm{RS}(T)} s^{\mathrm{ides}(\sigma)}. \tag{4}$$

Figure 5 lists 8-André I permutations, 8-André II permutations and 8-simsun permutations of the same tree shape T, all of which have 4 inverse descents and share the descent set $\mathrm{Des} = \{1, 4, 6\}$.

The proof of Theorem 7 is divided into two parts, (a) and (b). It turns out that the proof of relation (a) reduces to investigating the shuffle of permutations (see Section 3) and the proof of relation (b) proceeds by constructing a bijection between the set of n-André II permutations and the set of the n-simsun permutations (see Section 4).

3. A more general result on permutations

The main objective of this section is to prove Theorem 2. Before proceeding, we first demonstrate how to derive relation (a) in Theorem 7 using Theorem 2. To this end, we begin by stating the following corollary, which is an immediate consequence of Theorem 2.

Corollary 8. *Let σ and $\hat{\sigma}$ be two permutations in $\mathfrak{S}_j$ such that* $\mathrm{ides}(\sigma) = \mathrm{ides}(\hat{\sigma}) = j'$, *and let τ and $\hat{\tau}$ be two permutations in $\mathfrak{S}_k$ such that* $\mathrm{ides}(\tau) = \mathrm{ides}(\hat{\tau}) = k'$. *Then*

$$\sum_{\mu \in \sigma \vartriangle \tau} t^{\mathrm{ides}(\mu)} = \sum_{\mu \in \hat{\sigma} \triangledown \hat{\tau}} t^{\mathrm{ides}(\mu)}.$$

We are ready to establish relation (a) in Theorem 7 using Corollary 8.

Proof of relation (a) in Theorem 7. We proceed by induction on n. For $n = 1$, relation (a) clearly holds. Assume that it holds for all $p < n$. We aim to show that it also holds for n. Let T be an unlabeled (rooted) binary tree in U_n^{RL} with the left subtree T^l and the right subtree T^r of the root, respectively. By the definition of André permutations, we have

$$\mathrm{And}^I(T) = \bigcup_{\sigma \in \mathrm{And}^I(T^l)} \bigcup_{\tau \in \mathrm{And}^I(T^r)} \sigma \vartriangle \tau$$

and

$$\mathrm{And}^{II}(T) = \bigcup_{\hat{\sigma} \in \mathrm{And}^{II}(T^l)} \bigcup_{\hat{\tau} \in \mathrm{And}^{II}(T^r)} \hat{\sigma} \triangledown \hat{\tau}.$$

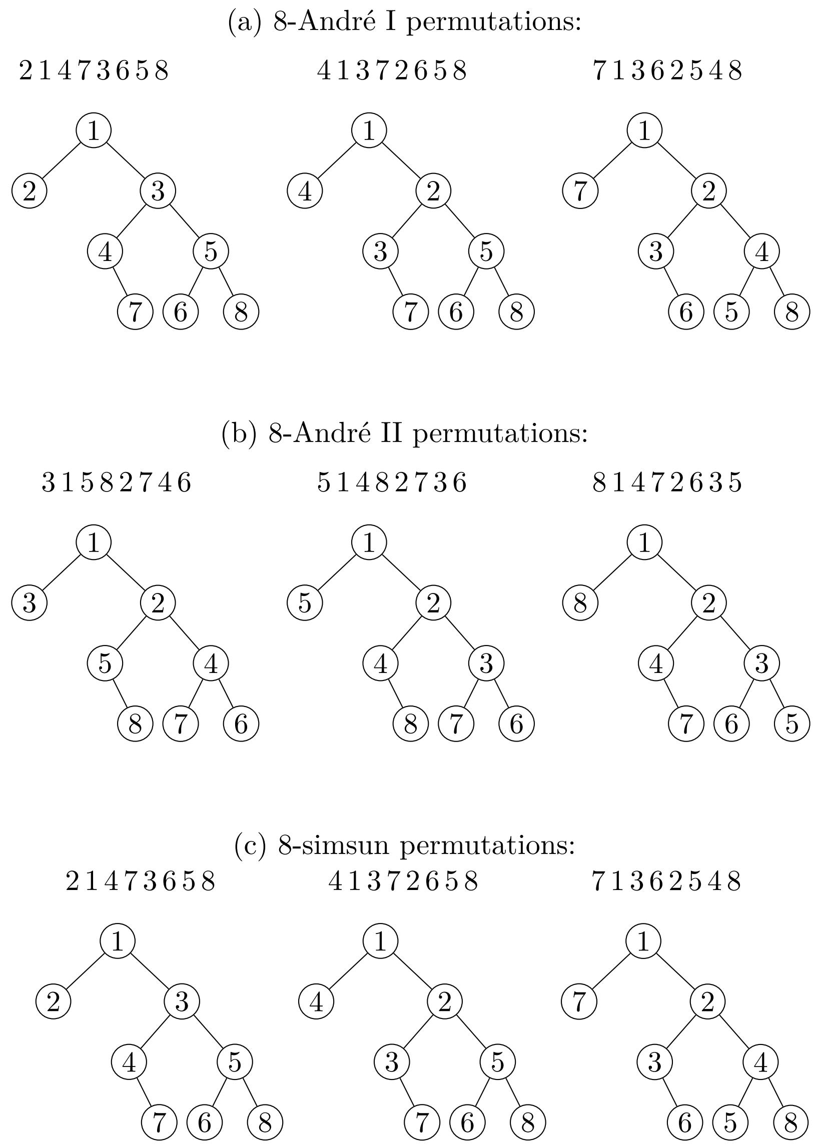

FIGURE 5. 8-André I permutations, 8-André I permutations and 8-simsun permutations with $\mathrm{Des} = \{1, 4, 6\}$ and $\mathrm{ides} = 4$

This implies that

$$\sum_{\mu \in \mathrm{And}^I(T)} t^{\mathrm{ides}(\mu)} = \sum_{\sigma \in \mathrm{And}^I(T^l)} \sum_{\tau \in \mathrm{And}^I(T^r)} \sum_{\mu \in \sigma \Delta \tau} t^{\mathrm{ides}(\mu)} \tag{5}$$

$$\sum_{\mu \in \mathrm{And}^{II}(T)} t^{\mathrm{ides}(\mu)} = \sum_{\hat{\sigma} \in \mathrm{And}^{II}(T^l)} \sum_{\hat{\tau} \in \mathrm{And}^{II}(T^r)} \sum_{\mu \in \hat{\sigma} \nabla \hat{\tau}} t^{\mathrm{ides}(\mu)}. \tag{6}$$

Note that T^l and T^r are the left and right subtrees of the root of T, so their vertices are less than n. The induction hypothesis implies that there exists a bijection ϕ^l between $\mathrm{And}^I(T^l)$ and $\mathrm{And}^{II}(T^l)$ such that for $\sigma \in \mathrm{And}^I(T^l)$ and $\phi(\sigma) \in \mathrm{And}^{II}(T^l)$, we have $\mathrm{ides}(\sigma) = \mathrm{ides}(\phi^l(\sigma))$. Similarly, there exists a bijection ϕ^r between $\mathrm{And}^I(T^r)$ and $\mathrm{And}^{II}(T^r)$ such that for $\tau \in \mathrm{And}^I(T^r)$ and $\phi^r(\tau) \in \mathrm{And}^{II}(T^r)$, we have $\mathrm{ides}(\tau) = \mathrm{ides}(\phi^r(\tau))$. Hence by Corollary 8, we arrive at

$$\sum_{\mu \in \sigma \triangle \tau} t^{\mathrm{ides}(\mu)} = \sum_{\mu \in \phi^l(\sigma) \triangledown \phi^r(\tau)} t^{\mathrm{ides}(\mu)}. \tag{7}$$

We therefore derive that

$$\begin{aligned}
\sum_{\mu \in \mathrm{And}^I(T)} t^{\mathrm{ides}(\mu)} &\overset{(5)}{=} \sum_{\sigma \in \mathrm{And}^I(T^l)} \sum_{\tau \in \mathrm{And}^I(T^r)} \sum_{\mu \in \sigma \triangle \tau} t^{\mathrm{ides}(\mu)} \\
&\overset{(7)}{=} \sum_{\phi^l(\sigma) \in \mathrm{And}^{II}(T^l)} \sum_{\phi^r(\tau) \in \mathrm{And}^{II}(T^r)} \sum_{\mu \in \phi^l(\sigma) \triangledown \phi^r(\tau)} t^{\mathrm{ides}(\mu)} \\
&\overset{(6)}{=} \sum_{\mu \in \mathrm{And}^{II}(T)} t^{\mathrm{ides}(\mu)}.
\end{aligned}$$

This confirms that relation (a) also holds for n. Thus, we complete the proof of Theorem 7 (a). □

We proceed to prove Theorem 2. As mentioned previously, the proof of Theorem 2 boils down to studying the shuffles of permutations. Let $\mathcal{S}_n$ denote the set of permutations of length n, where a permutation is defined as a sequence of n distinct integers (not necessarily restricted to $\{1, 2, \ldots, n\}$). Let $\pi \in \mathcal{S}_j$ and $\delta \in \mathcal{S}_k$ be two disjoint permutations, that is, permutations with no letters in common. We say that $\alpha \in \mathcal{S}_{j+k}$ is a shuffle of π and δ if both π and δ are subsequences of α. The set of shuffles of π and δ is denoted $\pi \sqcup \delta$. For example, let $\pi = 263$ and $\delta = 14$, we have $263 \sqcup 14 = \{26314, 26134, 26143, 21463, 21634, 21643, 12463, 14263, 12634, 12643\}$.

In his study of the theory of P-partitions, Stanley [26] established the following formula for the joint statistic (des, maj) over the set of permutation shuffles, which is referred to as Stanley's shuffle theorem. Bijective proofs were later found by Goulden [15], the second author and Zhang [20] and Stadler [24]. Recall that the Gaussian polynomial (also called the q-binomial coefficients) is given by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q^m)(1-q^{m-1})\cdots(1-q)}.$$

Theorem 9 (Stanley's Shuffle Theorem)**.** *Let $\pi \in \mathcal{S}_m$ and $\delta \in \mathcal{S}_n$ be two disjoint permutations, where* $\mathrm{des}(\pi) = r$ *and* $\mathrm{des}(\delta) = s$*. Then*

$$\sum_{\substack{\alpha \in \pi \sqcup \delta \\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix} m-r+s \\ k-r \end{bmatrix} \begin{bmatrix} n-s+r \\ k-s \end{bmatrix} q^{\mathrm{maj}(\pi)+\mathrm{maj}(\delta)+(k-s)(k-r)}.$$

To prove Theorem 2, it is necessary to introduce three special sets of shuffles, which are related to the sets $\sigma \Diamond \tau$ (resp. $\sigma \vartriangle \tau$, $\sigma \triangledown \tau$). Given two disjoint permutations $\pi = \pi_1 \cdots \pi_m \in \mathcal{S}_m$ and $\delta = \delta_1 \cdots \delta_n \in \mathcal{S}_n$,

- Let $\pi \sqcup_l \delta$ denote the set of shuffles $\alpha = \alpha_1 \cdots \alpha_{n+m}$ of π and δ such that $\alpha_1 = \delta_1$.
- Let $\pi \sqcup_{ls} \delta$ denote the set of shuffles $\alpha = \alpha_1 \cdots \alpha_{n+m}$ of π and δ such that $\alpha_1 = \delta_1$ and $\alpha_{n+m} = \delta_n$.
- Let $\pi \sqcup_{ll} \delta$ denote the set of shuffles $\alpha = \alpha_1 \cdots \alpha_{n+m}$ of π and δ such that $\alpha_1 = \delta_1$ and $\alpha_2 = \delta_2$.

For example, let $\pi = 263$ and $\delta = 14$, we have $263 \sqcup_l 14 = \{12463, 14263, 12634, 12643\}$, $263 \sqcup_{ls} 14 = \{12634\}$ and $263 \sqcup_{ll} 14 = \{14263\}$.

The following proposition establishes a connection between the set $\sigma \Diamond \tau$ (resp. $\sigma \vartriangle \tau$, $\sigma \triangledown \tau$) and the three special sets of shuffles introduced above.

Proposition 10. *For $\sigma \in \mathfrak{S}_j$ and $\tau \in \mathfrak{S}_k$, let $\sigma^{-1} = \sigma_1^{-1} \cdots \sigma_j^{-1}$ and $\tau^{-1} = \tau_1^{-1} \cdots \tau_k^{-1}$ denote the inverses of σ and τ respectively. Define $\pi = \sigma_1^{-1} \cdots \sigma_j^{-1}$ and $\delta = (j+1)(\tau_1^{-1}+j+1) \cdots (\tau_k^{-1}+j+1)$. Then*

$$\sum_{\mu \in \sigma \Diamond \tau} t^{\mathrm{ides}(\mu)} = \sum_{\alpha \in \pi \sqcup_l \delta} t^{\mathrm{des}(\alpha)}, \tag{8}$$

$$\sum_{\mu \in \sigma \vartriangle \tau} t^{\mathrm{ides}(\mu)} = \sum_{\alpha \in \pi \sqcup_{ls} \delta} t^{\mathrm{des}(\alpha)}, \tag{9}$$

$$\sum_{\mu \in \sigma \triangledown \tau} t^{\mathrm{ides}(\mu)} = \sum_{\alpha \in \pi \sqcup_{ll} \delta} t^{\mathrm{des}(\alpha)}. \tag{10}$$

Proof. Let $\mu = \sigma' 1 \tau' \in \sigma \Diamond \tau$. By definition, we see that $\mu \in \mathfrak{S}_{j+k+1}$. Thus, we may write $\mu = \mu_1 \cdots \mu_{j+k+1}$ as a permutation of the set $\{1, 2, \ldots, j+k+1\}$. Note that $\mu_{j+1} = 1$ and $\sigma' \cup \tau' = \{2, 3, \ldots, j+k+1\}$. Let the elements of σ' be $i_1 < i_2 < \cdots < i_j$ and let those of τ' be $j_1 < j_2 < \cdots < j_k$.

We now consider the inverse of μ. Assume that $\mu^{-1} = \mu_1^{-1} \cdots \mu_{j+k+1}^{-1}$. It is clear to see that $\mu_1^{-1} = j+1$ and

$$\pi = \sigma^{-1} = \mu_{i_1}^{-1} \mu_{i_2}^{-1} \cdots \mu_{i_j}^{-1}$$

and

$$\delta := (j+1)(\tau^{-1}+j+1) = \mu_1^{-1} \mu_{j_1}^{-1} \mu_{j_2}^{-1} \cdots \mu_{j_k}^{-1}.$$

Let $\alpha = \mu^{-1}$, Clearly, $\alpha \in \pi \sqcup_l \delta$ and $\mathrm{ides}(\mu) = \mathrm{des}(\mu^{-1}) = \mathrm{des}(\alpha)$. Moreover, this process is reversible. Thus, we prove (8).

For (9), suppose that $\mu = \sigma' 1 \tau' \in \sigma \bigtriangleup \tau$. By definition, $j+k+1 \in \tau'$, so $\mu^{-1}_{j+k+1} = \tau^{-1}_k + j + 1 = \delta_{k+1}$. Thus, $\alpha \in \pi \sqcup_{ls} \delta$, establishing (9).

Finally, if $\mu = \sigma' 1 \tau' \in \sigma \nabla \tau$, then $2 \in \tau'$, which implies $\mu^{-1}_2 = \tau^{-1}_1 + j + 1 = \delta_2$. Hence, $\alpha \in \pi \sqcup_{ll} \delta$, proving (10). □

For example, given $\sigma = 35124$ and $\tau = 231$, we see that $j = 5$, $\sigma^{-1} = 34152$ and $\tau^{-1} = 312$. Thus, $\pi = 34152$ and $\delta = 6978$.

(a) For $\mu = 692581473 \in \sigma \Diamond \tau$, we see that $\mu^{-1} = 639741852 \in \pi \sqcup_l \delta$.

(b) For $\mu = 682571493 \in \sigma \bigtriangleup \tau$, we see that $\mu^{-1} = 639741528 \in \pi \sqcup_{ls} \delta$.

(c) For $\mu = 693581472 \in \sigma \nabla \tau$, we see that $\mu^{-1} = 693741852 \in \pi \sqcup_{ll} \delta$.

With Proposition 10 at our disposal, the proof of Theorem 2 comes down to establishing the following assertion.

Theorem 11. *Assume that $\pi \in \mathcal{S}_j$ and $\delta \in \mathcal{S}_{k+1}$ are two disjoint permutations, where $\operatorname{des}(\pi) = j'$ and $\operatorname{des}(\delta) = k'$. Moreover, $\delta_1 < \delta_2$ and all of the elements of δ are larger than the elements of π. Then*

$$\sum_{\alpha \in \pi \sqcup_l \delta} t^{\operatorname{des}(\alpha)} = \sum_{i \geq 1} \binom{k+1-k'+j'}{i+j'-k'} \binom{j-j'+k'-1}{i-1} t^{i+j'},$$

$$\sum_{\alpha \in \pi \sqcup_{ls} \delta} t^{\operatorname{des}(\alpha)} = \sum_{i \geq 1} \binom{k-k'+j'}{i+j'-k'} \binom{j-j'+k'-1}{i-1} t^{i+j'},$$

$$\sum_{\alpha \in \pi \sqcup_{ll} \delta} t^{\operatorname{des}(\alpha)} = \sum_{i \geq 1} \binom{k-k'+j'}{i+j'-k'} \binom{j-j'+k'-1}{i-1} t^{i+j'}.$$

The following assertion can be viewed as refinements of Stanley's shuffle theorem, from which Theorem 11 follows immediately by letting $q \to 1$.

Theorem 12. *Assume that $\delta \in \mathcal{S}_m$ and $\pi \in \mathcal{S}_n$ are two disjoint permutations, where $\operatorname{des}(\delta) = r$ and $\operatorname{des}(\pi) = s$. Moreover, $\delta_1 < \delta_2$ and all of the elements of δ are larger than the elements of π. Then*

$$(1) \sum_{\substack{\alpha \in \pi \sqcup_l \delta \\ \operatorname{des}(\alpha)=d}} q^{\operatorname{maj}(\alpha)} = \begin{bmatrix} m-r+s \\ d-r \end{bmatrix} \begin{bmatrix} n-s+r-1 \\ d-s-1 \end{bmatrix} \times q^{\operatorname{maj}(\delta)+\operatorname{maj}(\pi)+(d-s)(d-r)},$$

$$(2) \sum_{\substack{\alpha \in \pi \sqcup_{ls} \delta \\ \operatorname{des}(\alpha)=d}} q^{\operatorname{maj}(\alpha)} = \begin{bmatrix} m-r+s-1 \\ d-r \end{bmatrix} \begin{bmatrix} n-s+r-1 \\ d-s-1 \end{bmatrix} \times q^{\operatorname{maj}(\delta)+\operatorname{maj}(\pi)+(d-s)(d-r)},$$

$$(3) \sum_{\substack{\alpha \in \pi \sqcup_{ll} \delta \\ \operatorname{des}(\alpha)=d}} q^{\operatorname{maj}(\alpha)} = \begin{bmatrix} m-r+s-1 \\ d-r \end{bmatrix} \begin{bmatrix} n-s+r-1 \\ d-s-1 \end{bmatrix} \times q^{\operatorname{maj}(\delta)+\operatorname{maj}(\pi)+(d-s+1)(d-r)}.$$

We conclude this section with a proof of the theorem. It turns out that the bijection used to establish Stanley's shuffle theorem by the second author and Zhang [20] plays a crucial role. Let $\mathcal{P}_n(t,m)$ denote the set of partitions $\lambda=(\lambda_1,\ldots,\lambda_n)$ such that $\lambda_n \geq t$ and $\lambda_1 \leq m$. We have

$$q^{nt}\begin{bmatrix} n+m-t \\ n \end{bmatrix} = \sum_{\lambda\in\mathcal{P}_n(t,m)} q^{|\lambda|}. \tag{11}$$

Thus, Stanley's shuffle theorem 9 is equivalent to the following statement.

Theorem 13. [20, Theorem 3.1] *Assume that $\delta \in \mathcal{S}_m$ and $\pi \in \mathcal{S}_n$ are two disjoint permutations, where $\mathrm{des}(\delta)=r$ and $\mathrm{des}(\pi)=s$. Let $\mathfrak{S}(\delta,\pi|d)$ denote the set of all shuffles of δ and π with d descents. Then there is a bijection Φ between $\mathfrak{S}(\delta,\pi|d)$ and $\mathcal{P}_{d-r}(d-s,m)\times\mathcal{P}_{n-d+r}(0,d-s)$, namely, for $\alpha\in\mathfrak{S}(\delta,\pi|d)$, we have $(\lambda,\mu)=\Phi(\alpha)\in\mathcal{P}_{d-r}(d-s,m)\times\mathcal{P}_{n-d+r}(0,d-s)$ such that*

$$\mathrm{maj}(\alpha)=|\lambda|+|\mu|+\mathrm{maj}(\delta)+\mathrm{maj}(\pi).$$

The following map is a desired bijection in Theorem 13, see [20, Lemma 3.5 and Lemma 3.7].

Definition 14 (The map Φ)**.** *Let $\delta=\delta_1\cdots\delta_m$ be a permutation with r descents and let $\pi=\pi_1\cdots\pi_n$ be a permutation with s descents. Assume that $\alpha=\alpha_1\cdots\alpha_{m+n}$ is the shuffle of δ and π with d descents. The pair of partitions $(\lambda,\mu)=\Phi(\alpha)$ can be constructed as follows: Let $\alpha^{(i)}$ denote the permutation obtained by removing $\pi_1,\pi_2,\ldots,\pi_i$ from α. Obviously, $\alpha^{(n)}=\delta$. Here we assume that $\alpha^{(0)}=\alpha$. For $1\leq i\leq n$, define*

$$t(i)=\mathrm{maj}(\alpha^{(i-1)})-\mathrm{maj}(\alpha^{(i)})-d_i(\pi),$$

where $d_i(\pi)$ denotes the number of descents in π greater than or equal to i.

Since there are d descents in α and there are r descents in δ, it follows that there exists $d-r$ permutations in $\alpha^{(1)},\ldots,\alpha^{(n)}$, denoted by $\alpha^{(i_1)},\ldots,\alpha^{(i_{d-r})}$ where $1\leq i_1<i_2<\cdots<i_{d-r}\leq n$, such that $\mathrm{des}(\alpha^{(i_l-1)})=\mathrm{des}(\alpha^{(i_l)})+1$ for $1\leq l\leq d-r$. Let $\{j_1,\ldots,j_{n-d+r}\}\in\{1,\ldots,n\}\setminus\{i_1,i_2,\ldots,i_{d-r}\}$, where $1\leq j_1<j_2<\cdots<j_{n-d+r}\leq n$. Then $\mathrm{des}(\alpha^{(j_l-1)})=\mathrm{des}(\alpha^{(j_l)})$ for $1\leq l\leq n-d+r$. The pair of partitions $(\lambda,\mu)=\Phi(\alpha)$ is defined by

$$\lambda=\big(t(i_{d-r}),t(i_{d-r-1}),\ldots,t(i_1)\big),$$

and

$$\mu=\big(t(j_1),t(j_2),\ldots,t(j_{n-d+r})\big).$$

More precisely,

$$m\geq t(i_{d-r})\geq\cdots\geq t(i_1)\geq d-s\geq t(j_1)\geq\cdots\geq t(j_{n-d+r})\geq 0.$$

Let $\sigma=\sigma_1\cdots\sigma_n\in\mathcal{S}_n$ and $r\notin\sigma$. Recall that $\sigma^{(i)}(r)$ denotes the permutation obtained by inserting r before σ_{i+1} (or after σ_i if $i=n$). For $0\leq i\leq n$, define the major increment

$$\mathrm{im}(\sigma,i,r)=\mathrm{maj}(\sigma^{(i)}(r))-\mathrm{maj}(\sigma)$$

and the major increment sequence

$$\mathrm{MIS}(\sigma, r) = (\mathrm{im}(\sigma, 0, r), \ldots, \mathrm{im}(\sigma, n, r)).$$

The first i elements of $\mathrm{MIS}(\sigma, r)$ is defined by

$$\mathrm{MIS}_i(\sigma, r) = (\mathrm{im}(\sigma, 0, r), \ldots, \mathrm{im}(\sigma, i-1, r)).$$

The proof of Theorem 13 relies on the following two propositions, which are also essential for proving Theorem 12.

Proposition 15. [20, Corollary 3.3] *Let $\sigma \in \mathcal{S}_n$ with k descents and $r \notin \sigma$. Then $\mathrm{MIS}(\sigma, r)$ is a shuffling of $k+1, k+2, \ldots, n$ and $k, \ldots, 1, 0$. In particular, if $\mathrm{des}(\sigma^{(i)}(r)) = \mathrm{des}(\sigma) + 1$, then*

$$\mathrm{im}(\sigma, i, r) = \max\{\mathrm{im}(\sigma, 0, r), \ldots, \mathrm{im}(\sigma, i-1, r)\} + 1,$$

otherwise,

$$\mathrm{im}(\sigma, i, r) = \min\{\mathrm{im}(\sigma, 0, r), \ldots, \mathrm{im}(\sigma, i-1, r)\} - 1.$$

Proposition 16. [20, Proposition 3.4] *Suppose that σ is a permutation of length m with r descents. Let $p, q \notin \sigma$ and let $\sigma^{(i-1)}(p)$ be the permutation by inserting p before σ_i. Then $\mathrm{MIS}_i(\sigma^{(i-1)}(p), q)$ is a permutation of the set $\{\mathrm{im}(\sigma, j, p) + \chi(q > p) \mid 0 \le j < i\}$, where $\chi(T) = 1$ if the statement T is true and $\chi(T) = 0$ otherwise.*

We are now in a position to prove Theorem 12.

Proof of Theorem 12. (1) Let $\alpha \in \pi \sqcup_l \delta$, where $\mathrm{des}(\alpha) = d$. Assume that $\Phi(\alpha) = (\lambda, \mu)$. To prove (1) in this theorem, it is equivalent to show that

$$m \ge \lambda_1 \ge \cdots \ge \lambda_{d-r} \ge d - s > \mu_1 \ge \cdots \ge \mu_{n-d+r} \ge 0.$$

From Theorem 13, we have

$$m \ge \lambda_1 \ge \cdots \ge \lambda_{d-r} \ge d - s \ge \mu_1 \ge \cdots \ge \mu_{n-d+r} \ge 0.$$

We proceed to show that $\mu_1 < d - s$ if $\mu \ne \emptyset$. Since $\alpha \in \pi \sqcup_l \delta$, we have $\alpha_1 = \delta_1$. Recall that $\alpha^{(1)}$ is the permutation obtained by removing π_1 from α. Then $\mathrm{des}(\alpha^{(1)}) = d - 1$ or $\mathrm{des}(\alpha^{(1)}) = d$.

Case 1.1. If $\mathrm{des}(\alpha^{(1)}) = d$, then $\mathrm{des}(\alpha^{(0)}) = \mathrm{des}(\alpha^{(1)}) = d$, and so $j_1 = 1$. Since $\alpha_1 = \delta_1$, it implies that π_1 could not be inserted in the first position. Moreover, $\mathrm{im}(\alpha^{(1)}, 0, \pi_1) = d$, so by Proposition 15, we derive that

$$\mathrm{maj}(\alpha^{(0)}) - \mathrm{maj}(\alpha^{(1)}) \le d - 1.$$

Thus, $\mu_1 = t(1) = \mathrm{maj}(\alpha^{(0)}) - \mathrm{maj}(\alpha^{(1)}) - d_1(\pi) \le d - 1 - s$.

Case 1.2. If $\mathrm{des}(\alpha^{(1)}) = d - 1$, then by the definition of the map Φ, we see that $j_1 > 1$, and $\mathrm{des}(\alpha^{(j_1)}) = d - j_1 + 1$. Similarly, π_j could not be inserted in the first position of $\alpha^{(j_1)}$ and $\mathrm{im}(\alpha^{(j_1)}, 0, \pi_{j_1}) = d - j_1 + 1$, so by Proposition 15, we derive that

$$\mathrm{maj}(\alpha^{(j_1-1)}) - \mathrm{maj}(\alpha^{(j_1)}) \le d - j_1.$$

By definition, $d_{j_1}(\pi) \geq s-j_1+1$ so $\mu_1 = t(j_1) = \mathrm{maj}(\alpha^{(j_1-1)}) - \mathrm{maj}(\alpha^{(j_1)}) - d_{j_1}(\pi) \leq d - j_1 - s + j_1 - 1 = d - s - 1$.

Conversely, let $\lambda \in \mathcal{P}_{d-r}(d-s, m)$ and $\mu \in \mathcal{P}_{n-d+r}(0, d-s-1)$. Assume that $\Phi^{-1}(\lambda, \mu) = \overline{\alpha}$, where the map Φ^{-1} is the inverse of Φ. In light of Theorem 13, we derive that $\overline{\alpha} = \overline{\alpha}_1 \cdots \overline{\alpha}_{n+m}$ is a shuffle of δ and π with d descents. To prove that $\overline{\alpha} \in \pi \sqcup_l \delta$, it suffices to show that $\overline{\alpha}_1 = \delta_1$. Suppose to the contrary that $\overline{\alpha}_1 = \delta_1$, that is, $\overline{\alpha}_1 = \pi_1$. We have $\Phi(\overline{\alpha}) = \Phi(\Phi^{-1}(\lambda, \mu)) = (\lambda, \mu)$. Let $\overline{\alpha}^{(1)}$ denote the permutation obtained by removing π_1 from $\overline{\alpha}$. Then $\mathrm{des}(\overline{\alpha}^{(1)}) = d-1$ or $\mathrm{des}(\overline{\alpha}^{(1)}) = d$.

Case 1.1'. If $\mathrm{des}(\overline{\alpha}^{(1)}) = d$, then $\mathrm{des}(\overline{\alpha}^{(0)}) = \mathrm{des}(\overline{\alpha}^{(1)}) = d$, and so $j_1 = 1$, and $d_1(\pi) = s$. Since $\overline{\alpha}_1 = \pi_1$, we derive that $\mathrm{maj}(\overline{\alpha}^{(0)}) - \mathrm{maj}(\overline{\alpha}^{(1)}) = d$. Thus,

$$\mu_1 = t(1) = \mathrm{maj}(\overline{\alpha}^{(0)}) - \mathrm{maj}(\overline{\alpha}^{(1)}) - d_1(\pi) = d - s.$$

Case 1.2'. If $\mathrm{des}(\overline{\alpha}^{(1)}) = d-1$, then by the definition of the map Φ, we see that $j_1 > 1$, and $\mathrm{des}(\overline{\alpha}^{(j_1)}) = d - j_1 + 1$. Since $\overline{\alpha}_1 = \pi_1$ and all the elements of δ are larger than the elements of π, it follows that $\overline{\alpha}_1^{(j_1-1)} = \pi_{j_1}$. Thus, $\mathrm{maj}(\overline{\alpha}^{(j_1-1)}) - \mathrm{maj}(\overline{\alpha}^{(j_1)}) = d - j_1 + 1$ and $d_{j_1}(\pi) = s - j_1 + 1$. Consequently,

$$\mu_1 = t(j_1) = \mathrm{maj}(\alpha^{(j_1-1)}) - \mathrm{maj}(\alpha^{(j_1)}) - d_{j_1}(\pi) = d - s.$$

In both cases, we derive that $\mu_1 = d - s$, which contradicts the condition that $\mu_1 \leq d - s - 1$. Therefore, the assumption is false, so $\overline{\alpha}_1 = \delta_1$, which implies that $\overline{\alpha} \in \pi \sqcup_l \delta$.

(2) Let $\alpha \in \pi \sqcup_{ls} \delta$, where $\mathrm{des}(\alpha) = d$. Assume that $\Phi(\alpha) = (\lambda, \mu)$. The proof of (2) in this theorem is equivalent to showing that

$$m > \lambda_1 \geq \cdots \geq \lambda_{d-r} \geq d - s > \mu_1 \geq \cdots \geq \mu_{n-d+r} \geq 0. \tag{12}$$

Observe that if $\alpha \in \pi \sqcup_{ls} \delta$, then $\alpha \in \pi \sqcup_l \delta$. According to (1) in this theorem, we see that

$$m \geq \lambda_1 \geq \cdots \geq \lambda_{d-r} \geq d - s > \mu_1 \geq \cdots \geq \mu_{n-d+r} \geq 0.$$

It remains to show that $\lambda_1 < m$ if $\lambda \neq \emptyset$.

Since $\alpha \in \pi \sqcup_{ls} \delta$, we have $\alpha_1 = \delta_1$ and $\alpha_{n+m} = \delta_m$. Recall that $\alpha^{(i)}$ is the permutation obtained by removing $\pi_1, \ldots, \pi_i$ from α. Observe that $\alpha^{(n)} = \delta$ and $\mathrm{des}(\delta) = r$, so $\mathrm{des}(\alpha^{(n-1)}) = r + 1$ or $\mathrm{des}(\alpha^{(n-1)}) = r$.

Case 2.1. If $\mathrm{des}(\alpha^{(n-1)}) = r + 1$, then $i_{d-r} = n$, and $d_n(\pi) = 0$. By definition, it is easy to see that $\mathrm{im}(\alpha^{(n)}, m, \pi_n) = m$. Since $\alpha_{n+m} = \delta_m$, it implies that π_n could not be inserted in the last position, by Proposition 15, we derive that $\mathrm{im}(\alpha^{(n)}, i, \pi_n) \leq m - 1$ for $0 \leq i < m$. Hence

$$\mathrm{maj}(\alpha^{(n-1)}) - \mathrm{maj}(\alpha^{(n)}) \leq m - 1. \tag{13}$$

It follows that

$$\lambda_1 = t(i_{d-r}) = \mathrm{maj}(\alpha^{(n-1)}) - \mathrm{maj}(\alpha^{(n)}) - d_n(\pi) \leq m - 1.$$

Case 2.2. If $\mathrm{des}(\alpha^{(n-1)}) = r$, then by the definition of the map Φ, we see that $i_{d-r} < n$. Moreover, $\mathrm{des}(\alpha^{(i_{d-r}-1)}) = r+1$ and $\mathrm{des}(\alpha^{(l)}) = r$ for $i_{d-r} \le l \le n$. According to Proposition 16 and by (13), we derive that

$$\mathrm{maj}(\alpha^{(i_{d-r}-1)}) - \mathrm{maj}(\alpha^{(i_{d-r})}) \le m - 1 + d_{i_{d-r}}(\pi)$$

It follows that

$$\lambda_1 = t(i_{d-r}) = \mathrm{maj}(\alpha^{(i_{d-r}-1)}) - \mathrm{maj}(\alpha^{(i_{d-r})}) - d_{i_{d-r}}(\pi) \le m-1.$$

Conversely, let (λ, μ) be a pair of partitions satisfying (12). Assume that $\Phi^{-1}(\lambda,\mu) = \overline{\alpha}$, where the map Φ^{-1} is the inverse of Φ. In light of the first result in this theorem, we derive that $\overline{\alpha} = \overline{\alpha}_1 \cdots \overline{\alpha}_{n+m} \in \pi \sqcup_l \delta$. To prove that $\overline{\alpha} \in \pi \sqcup_{ls} \delta$, it suffices to show that $\overline{\alpha}_{n+m} = \delta_m$. Suppose to the contrary that $\overline{\alpha}_{n+m} = \delta_m$ and assume that $\overline{\alpha}_{n+m} = \pi_n$. We have $\Phi(\overline{\alpha}) = \Phi(\Phi^{-1}(\lambda,\mu)) = (\lambda,\mu)$. Let $\overline{\alpha}^{(i)}$ denote the permutation obtained by removing $\pi_1, \ldots, \pi_i$ from $\overline{\alpha}$. Since $\overline{\alpha}_{n+m} = \pi_n$ and all of the elements of δ are larger than the elements of π, we derive that $\mathrm{des}(\overline{\alpha}^{(n-1)}) = r+1$. In this case, we see that $i_{d-r} = n$, and $d_n(\pi) = 0$. Since $\overline{\alpha}_{n+m} = \pi_n$, we derive that $\mathrm{maj}(\overline{\alpha}^{(n-1)}) - \mathrm{maj}(\overline{\alpha}^{(n)}) = m$. Thus,

$$\lambda_1 = t(i_{d-r}) = \mathrm{maj}(\overline{\alpha}^{(n-1)}) - \mathrm{maj}(\overline{\alpha}^{(n)}) - d_n(\pi) = m,$$

which contradicts the condition that $\lambda_1 < m$. Therefore, the assumption is false, so $\overline{\alpha}_{m+n} = \delta_m$, which implies that $\alpha \in \pi \sqcup_{ls} \delta$.

(3) Let $\alpha \in \pi \sqcup_{ll} \delta$, where $\mathrm{des}(\alpha) = d$. Assume that $\Phi(\alpha) = (\lambda, \mu)$. To establish (3) in this theorem, by means of (11), it is enough to show that

$$m \ge \lambda_1 \ge \cdots \ge \lambda_{d-r} > d-s > \mu_1 \ge \cdots \ge \mu_{n-d+r} \ge 0. \tag{14}$$

Note that if $\alpha \in \pi \sqcup_{ll} \delta$, then $\alpha \in \pi \sqcup_l \delta$. From the first part of this theorem, we derive that

$$m \ge \lambda_1 \ge \cdots \ge \lambda_{d-r} \ge d-s > \mu_1 \ge \cdots \ge \mu_{n-d+r} \ge 0.$$

We proceed to show that $\lambda_{d-r} > d-s$ if $\lambda \ne \emptyset$.

Since $\alpha \in \pi \sqcup_{ll} \delta$, we have $\alpha_1 = \delta_1$ and $\alpha_2 = \delta_2$. Recall that $\alpha^{(1)}$ is the permutation obtained by removing π_1 from α. Then $\mathrm{des}(\alpha^{(1)}) = d-1$ or $\mathrm{des}(\alpha^{(1)}) = d$.

Case 3.1. If $\mathrm{des}(\alpha^{(1)}) = d-1$, then by the definition of the map Φ, we see that $i_1 = 1$ and $d_1(\pi) = s$. Since $\alpha_1 = \delta_1 < \alpha_2 = \delta_2$, it implies that π_1 could not be inserted in the first position and the second position. Moreover, $\mathrm{im}(\alpha^{(1)}, 0, \pi_1) = d-1$ and $\mathrm{im}(\alpha^{(1)}, 1, \pi_1) = d$. Thus, by Proposition 15, we derive that $\mathrm{maj}(\alpha^{(0)}) - \mathrm{maj}(\alpha^{(1)}) > d$, as $\mathrm{des}(\alpha^{(0)}) = \mathrm{des}(\alpha^{(1)}) + 1$. It follows that

$$\lambda_{d-r} = t(i_1) = \mathrm{maj}(\alpha^{(0)}) - \mathrm{maj}(\alpha^{(1)}) - d_1(\pi) > d-s.$$

Case 3.2. If $\mathrm{des}(\alpha^{(1)}) = d$, then by the definition of the map Φ, we see that $i_1 > 1$. Moreover, $\mathrm{des}(\alpha^{(i_1)}) = d-1$ and $\mathrm{des}(\alpha^{(l)}) = d$ for $1 \le l \le i_1 - 1$. Since $\alpha_1 = \delta_1 < \alpha_2 = \delta_2$, it implies that π_{i_1} could not be inserted in the first position and the second position of $\alpha^{(i_1)}$. Moreover, $\alpha_1^{(i_1)} = \delta_1$ and $\alpha_1^{(i_1)} = \delta_2$, thus, we have $\mathrm{im}(\alpha^{(i_1)}, 0, \pi_{i_1}) = d-1$ and $\mathrm{im}(\alpha^{(i_1)}, 1, \pi_1) = d$. Hence $\mathrm{maj}(\alpha^{(i_1-1)}) - \mathrm{maj}(\alpha^{(i_1)}) > d$ since $\mathrm{des}(\alpha^{(i_1-1)}) = \mathrm{des}(\alpha^{(i_1)}) + 1$. Observe that $d_{i_1}(\pi) \le \mathrm{des}(\pi) = s$, it follows that

$$\mu_1 = t(i_1) = \mathrm{maj}(\alpha^{(i_1-1)}) - \mathrm{maj}(\alpha^{(i_1)}) - d_{i_1}(\pi) \ge d + 1 - s.$$

Conversely, let (λ, μ) be a pair of partitions satisfying (14). Assume that $\Phi^{-1}(\lambda, \mu) = \overline{\alpha}$, where the map Φ^{-1} is the inverse of Φ. In light of the first part of this theorem, we derive that $\overline{\alpha} = \overline{\alpha}_1 \cdots \overline{\alpha}_{n+m} \in \pi \sqcup_l \delta$. To prove that $\overline{\alpha} \in \pi \sqcup_{ll} \delta$, it suffices to show that $\overline{\alpha}_2 = \delta_2$. Suppose to the contrary and assume that $\overline{\alpha}_2 = \pi_1$. We have $\Phi(\overline{\alpha}) = \Phi(\Phi^{-1}(\lambda, \mu)) = (\lambda, \mu)$. Let $\overline{\alpha}^{(1)}$ denote the permutation obtained by removing π_1 from $\overline{\alpha}$. Then $\mathrm{des}(\overline{\alpha}^{(1)}) = d-1$ or $\mathrm{des}(\overline{\alpha}^{(1)}) = d$.

Case 3.1'. If $\mathrm{des}(\overline{\alpha}^{(1)}) = d-1$, then by the definition of the map Φ, we see that $i_1 = 1$. Since $\overline{\alpha}_2 = \pi_1$, it follows that $\mathrm{maj}(\overline{\alpha}^{(0)}) - \mathrm{maj}(\overline{\alpha}^{(1)}) = d$. Consequently,

$$\lambda_{d-r} = t(1) = \mathrm{maj}(\alpha^{(0)}) - \mathrm{maj}(\alpha^{(1)}) - d_1(\pi) = d - s.$$

Case 3.2'. If $\mathrm{des}(\overline{\alpha}^{(1)}) = d$, then by the definition of the map Φ, we see that $i_1 > 1$, and $\mathrm{des}(\overline{\alpha}^{(i_1)}) = d-1$ and $\mathrm{des}(\overline{\alpha}^{(l)}) = d$ for $1 \le l \le i_1 - 1$. Since $\overline{\alpha}_1 = \delta_1$ and $\overline{\alpha}_2 = \pi_1$, it implies that π_{i_1} should be inserted in the second position of $\alpha^{(i_1)}$, that is, $\overline{\alpha}_1^{(i_1-1)} = \delta_1$ and $\overline{\alpha}_2^{(i_1-1)} = \pi_{i_1}$, otherwise, we could not get $\overline{\alpha}^{(i_1-1)}, \ldots, \overline{\alpha}^{(1)}$ so that $\mathrm{des}(\overline{\alpha}^{(l)}) = d$ for $1 \le l \le i_1 - 1$ and $\overline{\alpha}_2 = \pi_1$ since $\delta_1 < \delta_2$. Moreover, $d_{i_1}(\pi) = s$. Hence,

$$\mu_1 = t(i_1) = \mathrm{maj}(\overline{\alpha}^{(i_1-1)}) - \mathrm{maj}(\overline{\alpha}^{(i_1)}) - d_{i_1}(\pi) = d - s.$$

In both cases, we derive that $\lambda_{d-r} = d-s$, which contradicts the condition that $\lambda_{d-r} > d - s$. Therefore, the assumption is false, so $\overline{\alpha}_2 = \delta_2$, which implies that $\overline{\alpha} \in \pi \sqcup_{ll} \delta$.

4. SIMSUN PERMUTATIONS AND ANDRÉ II PERMUTATIONS

This section is dedicated to establishing relation (b) in Theorem 7 by constructing a bijection between the set of n-simsun permutations and the set of n-André II permutations with the same tree shape T.

Theorem 17. *For any unlabeled binary tree T in U_n^{RL}, there exists a bijection Ω between the set of n-simsun permutations in $\mathrm{RS}(T)$ and the set of n-André II permutations in $\mathrm{And}^{II}(T)$.*

Proof. Given an unlabeled binary tree T in U_n^{RL}, let σ be a simsun permutation in $\mathrm{RS}(T)$, we aim to define $\tau = \Omega(\sigma)$ belonging to $\mathrm{And}^{II}(T)$.

Let $\hat{T} := \Psi(\sigma)$ be the increasing binary tree corresponding to σ under the bijection Ψ defined in Definition 3 and let

$$R_{\hat{T}} = \{v_0 < v_1 < \cdots < v_{m-1} < v_m\} \tag{15}$$

be the set of vertices that don't belong to any left subtrees of $\hat{T}$. By the definition of simsun tree, we see that $v_0 = 1$ and $v_m = n$. Let $\bar{R}_{\hat{T}}$ be the set of vertices that don't belong to $R_{\hat{T}}$. Assume that

$$\bar{R}_{\hat{T}} = \{s_1 < \cdots < s_{n-m-1}\}. \tag{16}$$

Note that $R_{\hat{T}} \cup \bar{R}_{\hat{T}} = \{1, 2, \ldots, n\}$.

We then relabel the elements of $R_{\hat{T}}$ according to the permutation

$$\begin{pmatrix} v_0 = 1 & v_1 & v_2 & v_3 & \cdots & n \\ 1 & v_0 + 1 = 2 & v_1 + 1 & v_2 + 1 & \cdots & v_{m-1} + 1 \end{pmatrix}.$$

For other vertices that belong to $\bar{R}_{\hat{T}}$, we just add one to each of their values. By this operation, we obtain a new binary tree $\widetilde{T}$. From the above construction, we see that

$$R_{\widetilde{T}} = \{1 < 2 < v_1 + 1 < \cdots < v_{m-1} + 1\}, \tag{17}$$

and

$$\bar{R}_{\widetilde{T}} = \{s_1 + 1 < \cdots < s_{n-m-1} + 1\}. \tag{18}$$

It is straightforward to check that $R_{\widetilde{T}} \cup \bar{R}_{\widetilde{T}} = \{1, 2, \ldots, n\}$ and $\widetilde{T}$ is an increasing binary tree sharing the same tree shape as $\hat{T}$. Then τ is defined to be the permutation generated by the increasing binary tree $\widetilde{T}$ under the bijection Ψ, that is, $\tau = \Psi^{-1}(\widetilde{T})$.

For example, for the simsun permutation $\sigma = 21473658$, the corresponding increasing binary tree is shown in Fig. 6 (a). We have $R_{\hat{T}} = \{1, 3, 5, 8\}$. Relabel the elements of $R_{\hat{T}} = \{1, 3, 5, 8\}$ according to the permutation

$$\begin{pmatrix} 1 & 3 & 5 & 8 \\ 1 & 2 & 4 & 6 \end{pmatrix}$$

and add one to each of the vertices not belong to $R_{\hat{T}}$. We obtain the increasing binary tree $\widetilde{T}$ shown on Fig. 6 (b). Then $\tau = \Psi^{-1}(\widetilde{T}) = 31582746$.

We proceed to show that $\tau \in \mathrm{And}^{II}(T)$. It suffices to show that $\widetilde{T}$ satisfies the property of André II trees: for any internal vertex s in $\widetilde{T}$, the right subtree $\widetilde{T}_r(s)$ contains the vertex with the minimum label in $\widetilde{T}(s)$ excluding s itself. By definition, the minima of an empty subtree is defined as $+\infty$.

Since $\hat{T}$ is a simsun tree, by definition, for any internal vertex $s \in \bar{R}_{\hat{T}}$, the minimum label in its left subtree is always larger than the minimum label in its right subtree. Otherwise, removing vertices with labels larger than this minimum label would result in a tree violating the simsun tree property. Thus, from the above construction, we see that for any internal vertex $s \in \bar{R}_{\widetilde{T}}$, the right subtree $\widetilde{T}_r(s)$ contains the vertex with the minimum label in $\widetilde{T}(s)$ excluding s itself.

For any internal vertex $s \in R_{\widetilde{T}}$, we claim that the minimum label in its left subtree $(\widetilde{T}_l(s))$ is always larger than the minimum label in its right

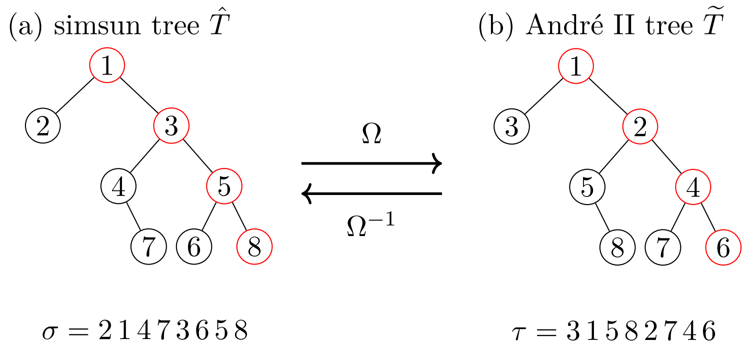

FIGURE 6. An illustration of the bijection Ω.

subtree ($\widetilde{T}_r(s)$). This holds because, in the increasing binary tree $\hat{T}$, the vertex with label equal to the minimum label of $\widetilde{T}_r(s)$ minus one is the parent of the vertex with label equal to the minimum label minus one in $\widetilde{T}_l(s)$. Consequently, the minimum label in $\widetilde{T}_l(s)$ is always larger than that in $\widetilde{T}_r(s)$, proving our claim. Hence for any internal vertex $s \in R_{\widetilde{T}}$, the right subtree $\widetilde{T}_r(s)$ contains the vertex with the minimum label in $\widetilde{T}(s)$ excluding s itself.

Thus, $\widetilde{T}$ is an André II tree, so $\tau = \Psi^{-1}(\widetilde{T})$ is an André II permutation with tree shape T. Moreover, it is easy to verify that the above procedure is reversible. Hence the map Ω is a bijection between $\mathrm{RS}(T)$ and $\mathrm{And}^{II}(T)$ for any given unlabeled binary tree T in U_n^{RL}. □

Remark 18. *The bijection Ω in Theorem 17 is constructed in the spirit of Scützenberger's jeu de taquin* [25]. *Lin and Kim* [21] *constructed a similar bijection between 0-1-2-increasing trees and binary increasing trees, which yields a bijection between 000-avoiding inversion sequences and Simsun permutations.*

From Theorem 17 and Proposition 5, we see that the bijection Ω preserves des and maj statistics. To establish relation (b) in Theorem 7, it remains to show that the bijection Ω preserves ides statistics. In fact, this bijection Ω also induces relations involving the inversions (inv), the imajor index (imaj) and right-to-left minima (RLmin) between simsun permutations and André II permutations (see Proposition 20). Recall that the number of inversions of $\sigma = \sigma_1 \cdots \sigma_n$, denoted $\mathrm{inv}(\sigma)$ is the count of pairs of indices (i, j) where $1 \le i < j \le n$ and $\sigma_i > \sigma_j$. The number of right-to-left minima, denoted $\mathrm{RLmin}(\sigma)$ is the count of the elements σ_i such that $\sigma_j > \sigma_i$ for every $j > i$. Let $\mathrm{IDes}(\sigma)$ denote the set of descents of the inverse permutation σ^{-1}, that is, $\mathrm{IDes}(\sigma) = \mathrm{Des}(\sigma^{-1})$. The imajor index of σ, denoted $\mathrm{imaj}(\sigma)$ is the sum of the indices in $\mathrm{IDes}(\sigma)$.

Before proceeding, let us first define an inversion and an idescent of an increasing binary tree.

Definition 19. *Let T be an increasing binary tree on the set $[n]$. An inversion of T is a pair of vertices (i,j), where $i > j$, and either j lies to the right of the path from root 1 to i or j is on the path from root 1 to i and left child of j is contained in this path. Moreover, $i-1$ is called an idescent of T if $(i,i-1)$ is an inversion of T.*

The number of inversions of T is denoted by inv (T), the set of idescents by $\mathrm{IDes}(T)$ and the set of vertices of T that don't belong to any left subtrees of T by R_T. For example, for the increasing binary tree $\hat{T}$ depicted in Fig. 6 (a), we see that $(2,1)$, $(4,3)$, $(7,3)$, $(7,6)$, $(7,5)$, $(6,5)$ are the inversions of $\hat{T}$, whereas, 1, 3, 6, 5 are idescents of $\hat{T}$. Thus, $\mathrm{inv}(\hat{T}) = 6$, $\mathrm{IDes}(\hat{T}) = \{1,\ 3,\ 5,\ 6\}$ and $R_{\hat{T}} = \{1,3,5,8\}$.

From the definition of Ψ, it is not difficult to derive the following proposition.

Proposition 20. *Let σ be a permutation and let $T_\sigma = \Psi(\sigma)$ be the increasing binary tree corresponding to σ under the bijection Ψ. Then*

$$\mathrm{inv}(\sigma) = \mathrm{inv}(T_\sigma), \quad \mathrm{RLmin}(\sigma) = |R_{T_\sigma}|, \quad \mathrm{IDes}(\sigma) = \mathrm{IDes}(T_\sigma).$$

We conclude this paper with the proof of the following proposition, which directly implies relation (b) in Theorem 7.

Proposition 21. *Given an unlabeled binary tree $T \in \mathrm{U}_n^{RL}$, let $\sigma \in \mathrm{RS}(T)$ be a simsun permutation and let $\tau = \Omega(\sigma) \in \mathrm{And}^{II}(T)$ be the corresponding André II permutation. Then*

$$\mathrm{ides}(\tau) = \mathrm{ides}(\sigma), \tag{19}$$

$$\mathrm{imaj}(\tau) = \mathrm{imaj}(\sigma) + \mathrm{ides}(\sigma), \tag{20}$$

$$\mathrm{inv}(\tau) = \mathrm{inv}(\sigma) + n - 1 - \mathrm{RLmin}(\sigma). \tag{21}$$

Proof. Let T be an increasing binary tree. Recall that R_T is the set of vertices of T that don't belong to any left subtrees of T. Let $\bar{R}_T$ be the set of vertices that don't belong to R_T. By definition, we see that for any $i \in R_T$, there doesn't exist j such that (i,j) is an inversion of T. Let

$$\begin{aligned}
A(T) :=& \{i-1 \colon (i,i-1) \text{ is an inversion, where } i \in \bar{R}_T, i-1 \in \bar{R}_T\},\\
B(T) :=& \{i-1 \colon (i,i-1) \text{ is an inversion, where } i \in \bar{R}_T, i-1 \in R_T\},\\
C(T) :=& \{(i,j) \colon (i,j) \text{ is an inversion of } T, \text{ where } i \in \bar{R}_T, j \in \bar{R}_T\},\\
D(T) :=& \{(i,j) \colon (i,j) \text{ is an inversion of } T, \text{ where } i \in \bar{R}_T, j \in R_T\}.
\end{aligned}$$

Suppose that $\hat{T}$ is the simsun tree corresponding to σ and $\widetilde{T}$ is the André II tree corresponding to τ, that is $\hat{T} = \Psi^{-1}(\sigma)$ and $\widetilde{T} = \Psi^{-1}(\tau)$. We then have

$$\mathrm{IDes}(\hat{T}) = A(\hat{T}) \cup B(\hat{T}) \quad \text{and} \quad \mathrm{inv}(\hat{T}) = |C(\hat{T})| + |D(\hat{T})|$$

and

$$\mathrm{IDes}(\widetilde{T}) = A(\widetilde{T}) \cup B(\widetilde{T}) \quad \text{and} \quad \mathrm{inv}(\widetilde{T}) = |C(\widetilde{T})| + |D(\widetilde{T})|.$$

From relations (15), (16), (17) and (18) in the construction of Φ, it is straightforward to derive that for $i \in [n]$, $i \in A(\hat{T}) \Leftrightarrow i+1 \in A(\widetilde{T})$ and $i \in B(\hat{T}) \Leftrightarrow i+1 \in B(\widetilde{T})$. It follows that for $i \in [n]$,

$$i \in \mathrm{IDes}(\hat{T}) \Leftrightarrow i+1 \in \mathrm{IDes}(\widetilde{T}).$$

Hence, by Proposition 20, we obtain (19) and (20).

Invoking relations (15), (16), (17) and (18) again, we derive that for $i \in [n]$ and $j \in [n]$,

$$(i,j) \in C(\hat{T}) \Leftrightarrow (i+1, j+1) \in C(\widetilde{T}). \tag{22}$$

On the other hand, for any $i \in \bar{R}_{\hat{T}}$, by the construction of the bijection Φ, we see that $i+1 \in \bar{R}_{\widetilde{T}}$. For a given $i \in \bar{R}_{\hat{T}}$, an inversion (i,j) in $\hat{T}$ with $j \in R_{\hat{T}}$ if and only if $(i+1, j+1)$ is an inversion in $\widetilde{T}$ with $j+1 \in R_{\widetilde{T}}$, and $i+1$ is not in the left subtree of $j+1$. Additionally, for each $i \in \bar{R}_{\hat{T}}$, there exists a $j \in R_{\widetilde{T}}$ such that $i+1$ lies in the left subtree of j and $(i+1, j)$ is an inversion in $\widetilde{T}$. Thus, for a given $i \in \bar{R}_{\hat{T}}$,

$$\begin{aligned} &| \{j \in R_{\widetilde{T}} : (i+1, j) \text{ is an inversion in } \widetilde{T}\} | \\ =&| \{j \in R_{\hat{T}} : (i, j) \text{ is an inversion in } \hat{T}\} | +1. \end{aligned}$$

It follows that

$$|D(\widetilde{T})| = |D(\hat{T})| + |\bar{R}_T|,$$

Combining this with (22), we obtain

$$\mathrm{inv}(\widetilde{T}) = \mathrm{inv}(\hat{T}) + |\bar{R}_T|,$$

and by Proposition 20, we derive that (21). This completes the proof. □

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